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Novel types of factorisable systems of differential equations

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Abstract. This paper considers systems of factorisable equations for different forms of the factorisation kernel $K(x, m)$. In one dimension some of these kernels lead only to trivial factorisable equations. However, in higher dimensions they yield some novel classes of factorisable equations. A partial explicit classification of these new factorisable systems is given.

1. Introduction

In a recent article [1] we considered the generalisation of the classical factorisation method [2, 3] to systems of second-order differential equations of the form

$$y''+R(x, m)y+\lambda y=0 \tag{1.1}$$

where $y\in R^n$ and $R(x, m)$ is an $n\times n$ matrix.

To factorise these systems one introduces the raising and lowering operators

$$H_m^+=K(x, m+1)-\frac{d}{dx}I \tag{1.2}$$

$$H_m^-=K(x, m)+\frac{d}{dx}I \tag{1.3}$$

which must then satisfy

$$H_{m+1}^-H_m^+y(\lambda, m)=[\lambda-L(m+1)]Iy(\lambda, m) \tag{1.4}$$

$$H_{m-1}^+H_m^-y(\lambda, m)=[\lambda-L(m)]Iy(\lambda, m) \tag{1.5}$$

where $K(x, m)$ is a $n\times n$ matrix, I is the $n\times n$ identity matrix and $L(m)$ is a scalar function.

To classify those systems which admit such a factorisation we considered in [1] (following the treatment of the scalar case in [2]) three forms for $K(x, m)$ and derived the equations which must be satisfied by them. These equations are

(1) $K=K_0(x)+mK_1(x)$

$$K_1'+K_1^2=-a^2I \tag{1.6a}$$

$$2K_0'+\{K_0, K_1\}=\begin{cases}-caI & a\neq 0\\ bI & a=0\end{cases} \tag{1.6b}$$

where $\{A, B\}=AB+BA$.

(2) $K = K_2(x)/m + K_0(x) + mK_1(x)$

$$K_2^2 = \gamma_1 I \tag{1.7a}$$

$$K_2' + \{K_2, K_0\} = \gamma_2 I \tag{1.7b}$$

$$2K_0' + \{K_1, K_0\} = \gamma_3 I \tag{1.7c}$$

$$2K_1' + K_1^2 = -a^2 I. \tag{1.7d}$$

(3) $K = K_0(x) + mK_1(x) + m^2 K_2(x)$

$$K_2^2 = \gamma_1 I \tag{1.8a}$$

$$2K_2' + 3\{K_1, K_2\} = \gamma_2 I \tag{1.8b}$$

$$K_1' + \{K_0, K_2\} + K_1^2 = \gamma_3 I \tag{1.8c}$$

$$2K_0' - \tfrac{1}{2}\{K_1, K_2\} + \{K_0, K_1\} = \gamma_4 I. \tag{1.8d}$$

The solution of these equations for $K(y, m)$ then yields the following formulae for $R(x, m)$ and $L(m)$:

$$R(x, m) = -[K^2(x, m) - K'(x, m) + L(m)I]$$

$$[L(m) - L(m+1)]I = K^2(x, m+1) - K^2(x, m) + K'(x, m+1) + K'(x, m).$$

In the scalar case ($n = 1$) it is easy to show that the system (1.8) has no non-trivial solutions, i.e. the only solutions are with K_2, K_1, K_0 being constants and the system (1.7) has a non-trivial solution only when $K_0 = 0$.

In [1] we therefore solved, in two dimensions, equation (1.6) in general and equation (1.7) for the special case $K_0 = 0$. We also showed (through examples), however, that the systems (1.7) and (1.8) have, for $n > 1$, non-trivial solutions that have no one-dimensional 'analogue'.

It is, therefore, our main objective in this paper to make a systematic study of these systems and give a fairly complete account of the solutions to (1.7) and (1.8) and some of the corresponding factorisable systems in two or more dimensions.

It turns out, however, that the systems (1.7) and (1.8) require a rather different treatment for the two cases $\gamma_1 \neq 0$ and $\gamma_1 = 0$. Accordingly we discuss these two cases in §§ 2 and 3, respectively. In § 4 we compute explicitly a few of the factorisable systems which are related to the solutions found in §§ 2 and 3.

As to the physical motivation for this study we wish to point out that recently the factorisation method was used to study supersymmetric models in quantum mechanics and elementary particles physics [4, 5]. Furthermore some recent generalisations of the factorisation method (in one dimension) proved to have new and interesting applications in atomic physics [6, 7]. Also, coupled systems of Schrödinger equations appear in the study of atomic systems and their interactions [8].

Thus although this paper does not present any explicit physical applications it is our hope that the factorisable systems derived here will ultimately contribute to the analytic study of some physical models. The classification of factorisable systems in several dimensions, especially those which have no one-dimensional analogues might be particularly useful in providing insights to physical systems which are generic to these higher dimensions.

2. Factorisations with $\gamma_1 \neq 0$

Lemma 1. For the system (1.7), if $\gamma_1 \neq 0$ then K_2 is a matrix with constant entries.

Proof. Multiplying (1.7*b*) from the left and right by K_2 and adding the results we obtain (using (1.7*a*))

$$(K_2'K_2 + K_2K_2') + 2K_2K_0K_2 + 2\gamma_1 K_0 = 2\gamma_2 K_2. \tag{2.1}$$

However, since (1.7*a*) implies that $K_2'K_2 + K_2'K_2 = 0$ we obtain, after multiplying (2.1) by K_2 on the right and using (1.7*a*), that

$$\{K_0, K_2\} = \gamma_2 I. \tag{2.2}$$

Subtracting (2.2) from (1.7*b*) we obtain $K_2' = 0$ which is the required result.

Lemma 2. For the system (1.8), if $\gamma_1 \neq 0$ then K_2 is a matrix with constant entries.

Proof. This result follows from the same steps of lemma 1 using equations (1.8*a*) and (1.8*b*).

We note, however, that even if K_2 is a constant matrix it is not obvious that the systems (1.7) and (1.8) cannot yield non-trivial factorisable systems (i.e. K_1 or K_0 are non-constant matrices). In fact, we already showed in [1] that if K_2 is a constant matrix and $K_0 = 0$ we still obtain non-trivial factorisations through the system (1.7). For the rest of this section we investigate, therefore, the existence of non-trivial solutions for the systems (1.7) and (1.8) when $\gamma_1 \neq 0$.

Lemma 3. If K_0, K_1, K_2 form a solution of (1.7) (or (1.8)) then $K_i = E^{-1}K_iE$, $i = 0, 1, 2$ (where E is a matrix with constant entries) is also a solution of this equation.

Proof. This is obvious.

We now observe that a matrix K which satisfies

$$K^2 = \gamma I \tag{2.3}$$

is similar to a matrix of the form

$$I_{m,n} = \gamma^{1/2}\left(\begin{array}{c|c} I_m & 0 \\ \hline 0 & -I_n \end{array}\right) \tag{2.4}$$

where I_m is the identity matrix in m dimensions. In fact, if M is the canonical Jordan form of K then (2.3) implies that $M^2 = \gamma I$ from which it is easy to infer the desired result.

In view of lemma 3 and the discussion above we now have the following corollary.

Corollary. If $\gamma_1 \neq 0$ then it is enough to solve (1.7) (or (1.8)) for K_2 in the form given by (2.4).

In fact any other solution will have to be similar to the general solution of the system which uses $I_{m,n}$ as K_2.

Theorem 1. If $\gamma_1 \neq 0$ then the only solution to the system (1.8) is the trivial solution where K_0, K_1, K_2 are matrices with constant entries.

Proof. Using $I_{m,n}$ as K_2 we first write K_0, K_1 in the corresponding block form

$$K_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad K_0 = \begin{pmatrix} R & S \\ P & Q \end{pmatrix}. \tag{2.5}$$

From (1.8*b*) we now infer

$$A = -D = \alpha I. \tag{2.6}$$

Moreover, from (1.8*c*) we obtain

$$B' = C' = 0 \tag{2.7}$$

and

$$2R + \alpha^2 I_m + BC = \gamma_3 I_m \tag{2.8}$$

$$-2Q + \alpha^2 I_n + CB = \gamma_3 I_n. \tag{2.9}$$

Hence B, C, R, Q are matrices with constant entries. Finally (1.8*d*) yields

$$S' + RB + BQ = 0 \tag{2.10}$$

$$P' + QC + CR = 0. \tag{2.11}$$

Substituting for R, Q in (2.10) and (2.11) from (2.8) and (2.9) we obtain

$$S' = P' = 0 \tag{2.12}$$

which completes the proof of this theorem.

As to the system (1.7) we already remarked that this system has a non-trivial solution with $K_0 = 0$. It is, therefore, of interest to determine under what conditions this system has a solution with $K_0 \neq 0$. While our first result in this direction is in n dimensions we consider for the rest of the discussion the system (1.7) in two dimensions only.

Lemma 4. If $\gamma_1 \neq 0$, $\gamma_2 \neq 0$ and K_2 is similar to $I_{m,0}$ then the only solution of (1.7) is the trivial solution.

Proof. From (1.7*b*) we infer that K_0 is a multiple of the identity matrix and hence from (1.7*c*) that K_1 a matrix with constant entries which proves our statement.

Theorem 2. In two dimensions if $\gamma_1 \neq 0$, $\gamma_2 \neq 0$ and K_2 is similar to $I_{1,1}$ then the only solution of (1.7) is the trivial one.

Proof. Letting K_1, K_0 be as in (2.5) (where the entires are numbers) we obtain from (1.7*b*)

$$R = -Q = \tfrac{1}{2}\gamma_2. \tag{2.13}$$

Furthermore, from (1.7*c*) we obtain

$$2S' + (A + D)S = 0 \qquad 2P' + (A + D)P = 0 \tag{2.14}$$

$$\gamma_2 A + CS + BP = \gamma_3 \qquad -\gamma_2 D + PB + CS = \gamma_3. \tag{2.15}$$

From (2.15) it follows that $A + D = 0$ and hence $S' = P' = 0$, i.e. S, P are constants. Furthermore, using the fact that $A + D = 0$ it is easy to see from (1.7*d*) that B, C, $A - D$ are also constants which complete the proof.

Theorem 3. In two dimensions if $\gamma_1 \neq 0$, $\gamma_2 = 0$, $\gamma_3 \neq 0$ then the only solution of the system (1.7) is the trivial solution.

Proof. The proof follows essentially the same steps as in theorem 2 and will be omitted here.

Theorem 4. In two dimensions if $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, $\gamma_3 = 0$ and K_2 is similar to $I_{1,1}$ then there exists a non-trivial solution of (1.7) with $K_0 \neq 0$.

Proof. From (1.7*b*) it follows that $R = Q = 0$ and from (1.7*c*) we obtain equation (2.14) and

$$BP + CS = 0. \tag{2.16}$$

Hence

$$S = d_2 J^{1/2} \qquad P = d_3 J^{1/2} \tag{2.17}$$

where $J = \exp(-\int(A + D)\,dx)$. Furthermore, from (1.7*d*) we infer that

$$A - D = c_1 J \qquad B = c_2 J \qquad C = c_3 J. \tag{2.18}$$

Hence the condition (2.16) is equivalent to

$$c_2 d_3 + c_3 d_2 = 0 \tag{2.19}$$

and this is the only constraint on the elements of K_0, K_1.

Thus, the general solution of (1.7) under the conditions of theorem 4 is

$$K_0 = J^{1/2}\begin{pmatrix} 0 & d_2 \\ d_3 & 0 \end{pmatrix}$$

and where K_1 is the general solution of (1.7*d*) (which was discussed in [1]) subject to the condition (2.19).

3. Factorisations with $\gamma_1 = 0$

Our major objective for the rest of this paper is to show the existence of non-trivial solutions to the systems (1.7) and (1.8) when $\gamma_1 = 0$. However, since the differential equations obtained for the matrix elements of K_i for $n > 2$ are rather intractable we shall consider only the two-dimensional case.

To begin with we observe that the general form of K_2 which satisfy the condition $K_2^2 = 0$ is

$$K_2 = \begin{pmatrix} q_1(x) & q_2(x) \\ q_3(x) & -q_1(x) \end{pmatrix} \qquad q_1^2 + q_2 q_3 = 0. \tag{3.1}$$

This can be rewritten, however, as

$$K_2 = q(x)\begin{pmatrix} \mathrm{i}p(x) & p^2(x) \\ 1 & -\mathrm{i}p(x) \end{pmatrix} = q(x)B. \tag{3.2}$$

Lemma 5. For the systems (1.7) and (1.8) if $\gamma_1 = 0$ then $p(x)$ is a constant.

Proof. From (1.7b) we obtain the following differential equations:

$$q' + q(R+Q) = 0 \tag{3.3}$$

$$(p^2 q)' + p^2 q(R+Q) = 0 \tag{3.4}$$

$$\mathrm{i}(pq)' + q(2\mathrm{i}pR + p^2 P + S) = \gamma_2 \tag{3.5}$$

$$-\mathrm{i}(pq)' + q(-2\mathrm{i}pQ + p^2 P + S) = \gamma_2. \tag{3.6}$$

Subtracting (3.6) from (3.5) we obtain

$$(pq)' + pq(R+Q) = 0. \tag{3.7}$$

Hence

$$pq = c_1 J_1 \qquad p^2 q = c_2 J_1 \tag{3.8}$$

where $J_1 = \exp(-\int (R+Q)\,\mathrm{d}x)$. Since $q \neq 0$ (otherwise $K_2 = 0$) it follows that p is a constant. The proof of our statement for the system (1.8) is similar.

Corollary. If $\gamma_1 = 0$ then K_2 is similar to

$$K_2 = q(x)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = q(x)A. \tag{3.9}$$

Proof. Since p is a constant we infer from the Jordan canonical form theorem that

$$B = E^{-1}AE$$

where E is a matrix with constant entries.

In view of this corollary and lemma 3 we can consider only those K_2 in the form given by equation (3.9).

Proposition 1. The system (1.7) in two dimensions admits non-trivial solutions with $\gamma_1 = \gamma_2 = 0$ and $K_0 \neq 0$.

Proof. Since we assume that $q(x) \neq 0$ it follows from (1.7b) that $S = 0$ and

$$q' + (R+Q)q = 0. \tag{3.10}$$

Moreover, we deduce from (1.7c) that $B(R+Q) = 0$. Hence, either $B = 0$ or $R + Q = 0$. When $B = 0$ we obtain from (1.7c) and (1.7d) the following equations for the matrix elements of K_1, K_0:

$$A' + A^2 = -a^2 \qquad D' + D^2 = -a^2 \tag{3.11}$$

$$R' + AR = \tfrac{1}{2}\gamma_3 \qquad Q' + DQ = \tfrac{1}{2}\gamma_3 \tag{3.12}$$

$$C' + C(A+D) = 0 \tag{3.13}$$

$$2P' + (A+D)P + C(R+Q) = 0. \tag{3.14}$$

These equations are consistent and independent of any additional constraint. Moreover, it is easy to solve this system in the order indicated above. In the second case where $R + Q = 0$, $B \neq 0$ we must require for consistency $A = D$ and $P = \gamma_3 = 0$. The differential equations which must be satisfied by A, B, R in this case are given by equation (4.5).

Proposition 2. The system (1.8) is two dimensions has non-trivial solutions with $\gamma_1 = \gamma_2 = 0$ and $K_2 \neq 0$.

We omit the proof (which is similar to the one above) but note that such a non-trivial solution is obtained under the present assumptions when $S = B = 0$ and the remaining elements of K_0, K_1, K_2 satisfy the following equations:

$$A' + A^2 = \gamma_3 \qquad D' + D^2 = \gamma_3 \tag{3.15}$$

$$R' + AR = \gamma_4 \qquad Q' + DQ = \gamma_4 \tag{3.16}$$

$$C' + C(A + D) = -A(R + Q) \tag{3.17}$$

$$2q' + 3q(A + D) = 0 \tag{3.18}$$

$$2P' + P(A + D) = \tfrac{1}{2}q(A + D) - C(R + Q). \tag{3.19}$$

Once again we note that this is a weakly coupled system whose general solution can be easily obtained when solved in the order given above.

4. Examples of some factorisable systems

Collecting the results of the previous two sections we compute here explicitly three classes of factorisable systems in two dimensions which have no one-dimensional analogues. These three classes correspond to those described by theorem 4 and propositions 1 and 2. We emphasise, however, that the list compiled here is far from being exhaustive. In fact our examples represents only some very special subsets of the possible factorisable systems described by these theorems.

4.1. SF-A

This class of special factorisable (SF) systems correspond to those described by theorem 4 (case 1). To satisfy the condition (2.1g) we let $c_2 = c_3 = 0$. Furthermore, we set $A = D$. The form of the matrices K_0, K_1 is then

$$K_1 = AI \qquad K_0 = J^{1/2}\begin{pmatrix} 0 & d_2 \\ d_3 & 0 \end{pmatrix} \tag{4.1}$$

while K_2 is given by (2.3). The equation which A satisfies is

$$A' + A^2 = a^2$$

(we replace here and in the following $-a^2$ by a^2 in (1.7d)) and

$$L(m) = -\left(\frac{\gamma_1}{m} + m^2 a^2\right). \tag{4.2}$$

Using the special solution $A = a$ for A we find that the corresponding form of

$$R(x, m) = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \tag{4.3}$$

is

$$R_{11}=-\frac{\exp(-2ax)}{m}\{m[2aq_1\exp(2ax)+d_2d_3]+(d_2q_3+d_2q_2)\exp(ax)\}$$

$$R_{12}=-a\exp(-ax)[2q_2\exp(ax)+d_2(2m+1)]$$

$$R_{21}=-a\exp(-ax)[2q_3\exp(ax)+d_3(2m+1)]$$

$$R_{22}=\frac{\exp(-2ax)}{m}\{m[2aq_1\exp(2ax)-d_2d_3]-(d_2q_3+d_3q_2)\exp(ax)\}.$$

4.2. SF-B

Using proposition 1 with $s=p=r+q=\gamma_3=0$, $A=D$ we deduce that the general form for $L(m)$ and the matrices K_0, K_1, K_2 is

$$L(m)=-a^2m^2 \tag{4.4}$$

$$K_0=\begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix} \qquad K_1=\begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \qquad K_2=bA \tag{4.5}$$

where b is a constant. The differential equations for A, B, R are

$$\begin{aligned} &A'+A^2=a^2 \qquad B'+2AB=0 \\ &R'+AR=0 \end{aligned} \tag{4.6}$$

i.e.

$$B=c_2J \qquad R=d_1J^{1/2}. \tag{4.7}$$

Using $A=a$ as a solution for A we obtain for $R(x, m)$

$$R_{11}=-\exp(-2ax)\{d_1[(2m+1)a\exp(ax)+d_1]+bc_2\}$$

$$R_{12}=-2m(m+1)ac_2\exp(-2ax)$$

$$R_{21}=-2ab$$

$$R_{22}=\exp(-2ax)\{d_1[(2m+1)a\exp(ax)-d_1]-bc_2\}.$$

4.3. SF-C

To simplify this case further we let

$$K_1=AI \qquad K_2=\begin{pmatrix} R & 0 \\ P & -R \end{pmatrix} \qquad K_2=q(x)A. \tag{4.8}$$

Solving the appropriate differential equations yields

$$\begin{aligned} &B=b_0J^{3/2} \qquad R=d_1J^{1/2} \\ &P=d_3J^{1/2}-\tfrac{1}{4}b_0J^{3/2}. \end{aligned} \tag{4.9}$$

Choosing $A=a$ as a solution for A and $L(m)=-a^2m^2$ we obtain for $R(x, m)$

$$R_{11}=-d_1\exp(-2ax)[(2m+1)a\exp(ax)+d_1]$$

$$R_{12}=0$$

$$R_{21}=\tfrac{1}{4}a(2m+1)\exp(-3ax)[4d_3\exp(2ax)+b_0(2m+3)(2m-1)]$$

$$R_{22}=d_1\exp(-2ax)[(2m+1)a\exp(ax)-d_1].$$

Although this system is separable we note that non-separable systems can be obtained using proposition 2 when we let $S\neq 0$.

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